Direct Path Method for Flexible Multibody Spacecraft Dynamics

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The dynamic equations of motion of a multibody flexible spacecraft with topological tree configuration are derived by the direct path method. The terminal bodies are treated as flexible and the interconnected bodies as rigid. Bodies are connected by gimbal hinges supplied with torsional springs and dampers. The three gimbal rotational axes may be either completely free or locked sequentially. Three approaches are suggested in this analysis, namely; the Lagrangian-Newtonian approach, the complete Newtonian approach, and the complete Lagrangian approach. The direct paths, the direct position vectors, and the incidence matrix are the key features of this analysis. They efficiently govern the contribution from the single body to the overall dynamic equations of the system. From the physical point of view, the complete Newtonian approach is the most straightforward. Torque equations are obtained from directly taking the moment about the hinge points rather than going through the lengthier differentiation of the angular momentum. This approach also gives better physical understanding to the interaction among motions of the individual bodies. From the mathematical point of view, the complete Lagrangian approach is the fastest. The lengthy elimination of the interacting forces and torques does not appear in this approach.

Nomenclature

$\mathfrak{D}^j$ = rate of change in damping energy dissipation of body j
$\boldsymbol{e}^e_\xi$ = unit vectors along x^e_ξ ($\xi = 1,2,3$)
$\boldsymbol{e}^j_\beta$ = unit vectors along x^j_β ($\beta = 1,2,3$)
$\boldsymbol{e}^r_\gamma$ = unit vectors along x^r_γ ($\gamma = 1,2,3$)
$\boldsymbol{e}^{jr}_\eta$ = base vectors along x^{jr}_η ($\eta = 1,2,3$)
$\boldsymbol{e}^{jr^*}_\zeta$ = reciprocal base vectors associated with $\boldsymbol{e}^{jr}_\eta$ ($\zeta = 1,2,3$)
$\boldsymbol{e}^{vg}_\lambda$ = base vectors along x^{vg}_λ ($\lambda = 1,2,3$)
$\boldsymbol{e}^{vg^*}_\mu$ = reciprocal base vectors associated with $\boldsymbol{e}^{vg}_\lambda$ ($\mu = 1,2,3$)
$\boldsymbol{f}^{je}$ = $f^{je}_\beta \, \boldsymbol{e}^j_\beta$ = force vector acting on the unit mass in body j due to the enviromental disturbances
$\boldsymbol{F}^j$ = $F^j_\beta \, \boldsymbol{e}^j_\beta$ = force vector acting on body j at O_j from its limb body
$\boldsymbol{g}^{v(j)}$ = $g^{v(j)}_\beta \, \boldsymbol{e}^v_\beta$ = direct position vector $\overline{O_v Q'_j}$
$\boldsymbol{G}^{jr}$ = $\boldsymbol{e}^j_\beta \, G^{jr}_{\beta\zeta} \, \boldsymbol{e}^{jr^*}_\zeta$ = Euler transformation tensor between $\boldsymbol{e}^j_\beta$ and $\boldsymbol{e}^{jr^*}_\zeta$
$\boldsymbol{G}^v$ = $\boldsymbol{e}^v_\beta \, G^v_{\beta\mu} \, \boldsymbol{e}^{vg^*}_\mu$ = gimbal transformation tensor between $\boldsymbol{e}^v_\beta$ and $\boldsymbol{e}^{vg^*}_\mu$
$\boldsymbol{G}^{u(T)}$ = $\boldsymbol{e}^{ug^*}_\mu \, G^{u(T)}_{\mu\beta} \, \boldsymbol{e}^u_\beta$ = transpose gimbal transformation tensor between $\boldsymbol{e}^{ug^*}_\mu$ and $\boldsymbol{e}^u_\beta$
$\boldsymbol{h}^{jm}$ = $h^{jm}_\beta \, \boldsymbol{e}^j_\beta$ = position vector $\overline{O_j O_m}$
$\mathcal{K}^j$ = kinetic energy of body j
K^j_{kl} = generalized stiffness matrix of body j ($k, = 1,2,\ldots n_j$)
K^{jg}_σ = torsional spring constants about the gimbal rotational axes between body j and its limb body ($\sigma = 1,2,3$)
N = total number of bodies
n_j = number of the natural vibration modes for body j
O_e = origin of the Earth coordinates
O_j = origin of the body j coordinates
O_r = origin of the spacecraft reference coordinates
P_j, P'_j = positions of the mass element dm^j before and after elastic deformation
Q'_j = instantaneous mass center of body j
$q^j_l(t)$ = time function associated with $\boldsymbol{\phi}^j_l(\boldsymbol{r}^j)$
$\boldsymbol{r}^j$ = $r^j_\beta \, \boldsymbol{e}^j_\beta$ = position vector $\overline{O_j P_j}$
$\boldsymbol{R}^j$ = $\boldsymbol{R}^j_\gamma \, \boldsymbol{e}^r_\gamma$ = position vector $\overline{O_r O_j}$
$\boldsymbol{R}^r$ = $R^r_\xi \, \boldsymbol{e}^e_\xi$ = position vector $\overline{O_e O_r}$
$\mathcal{S}^j$ = strain energy of body j
$\boldsymbol{T}^j$ = $T^j_\beta \, \boldsymbol{e}^j_\beta$ = torque vector acting on body j at O_j from its limb body
$\boldsymbol{u}^j$ = $u^j_\beta \, \boldsymbol{e}^j_\beta$ = displacement vector $\overline{P_j P'_j}$
V^j_{kl} = generalized damping matrix of body j ($k,l = 1,2,\ldots n_j$)
V^j_σ = torsional damping constants about the gimbal rotational axes between body j and its limb body ($\sigma = 1,2,3$)
x^e_ξ = Earth coordinates
x^j_β = body j coordinates
x^r_γ = spacecraft reference coordinates
x^{jr}_η = Euler rotational axes between x^r_γ and x^j_β
x^{vg}_λ = gimbal rotational axes between x^v_β and $x^{v^*}_\alpha$
$x^{v^*}_\alpha$ = nominal (mounting) coordinates of body v on its limb body
ϵ^{vj} = incidence matrix ($v,j = 1,2,\ldots N$)
ρ^j = $\rho^j_\xi \, \boldsymbol{e}^e_\xi$ = position vector $\overline{O_e P'_j}$
θ^{jr}_η = Euler angles about x^{jr}_η ($\eta = 1,2,3$)
θ^v_λ = gimbal rotational angles about x^{vg}_λ ($\lambda = 1,2,3$)
$\boldsymbol{\omega}^j$ = $\omega^j_\beta \, \boldsymbol{e}^j_\beta$ = angular velocity vector of body j
$\boldsymbol{\omega}^r$ = $\omega^r_\gamma \, \boldsymbol{e}^r_\gamma$ = angular velocity vector of the reference coordinates
$\boldsymbol{\phi}^j_l(\boldsymbol{r}^j)$ = $\phi^j_{l\beta} \, \boldsymbol{e}^j_\beta$ = natural vibration modal displacement vector functions of body j based on: "Point O_j is constrained from any linear or rotational displacements; and the rest of the boundary is free of force and torque" ($l = 1,2,\ldots n_j$)

I. Introduction

THIS paper formulates the dynamic equations of motion of a flexible multibody spacecraft with arbitrary topological tree configuration. The terminal bodies are

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treated as flexible and the interconnected bodies as rigid. Bodies are connected by gimbaled hinges supplied with torsional springs and dampers. The three gimbal rotational axes may be either completely free or locked sequentially. A general-purpose simulation computer program can be developed from this analysis.

Dynamic equations of motion of the multibody system have been investigated by many people in recent years.[1-17] Fletcher, Rongved, and Yu[1] first treated the simple case of two rigid bodies. Hooker and Margulies[2] derived the dynamic equations of the n-rigid-body system. Later, Hooker[3] presented the modification technique to eliminate the interacting constraint torques. In their papers, the concept of "barycenter" was introduced, which can be interpreted as the new mass center obtained by loading each joint of the body with the residual mass of the subsystem of bodies connected to that point. Roberson and Wittenburg[4] treated the n-rigid-body independently and derived the dynamic equations in matrix form. In this paper, the interacting torques were not eliminated.

Ness and Farrenkopf[5] and Ho and Bluck[6] extended the model into the flexible n-body system. Ness and Farrenkopf, using the flexible-terminal/rigid-interconnected model, chose the unified approach to deal with the nonlinear dynamic equations of the total motion of the system. Ho and Gluck, using the all-flexible model, chose the perturbation approach to deal with the linearized dynamic equations of the perturbed motion. They considered the total motion to be the superposition of a small perturbed motion about the nominal motion with all the flexible bodies are temporarily assumed to be rigid. In Refs. 5 and 6, the dynamic equations of a single body are first derived analytically. Then, they are combined with the equations of the adjacent body through a computerized inductive method following the combining algorithm. The interacting forces and torques are eliminated at each combining step. Modifications for the case of the locked gimbal axes are performed simultaneously. The drawback of the computerized inductive method is that the loading and reloading of the coefficient matrices in the dynamic equations of the subsystems of bodies will, undoubtedly, take up much computer time. And also it does not provide the analytical overall dynamic equations of the complete system.

The direct path method was originally developed in 1973.[4] At the same time Hooker[15] and Likins[16] independently developed their analysis using the barycenter approach. The differences between the direct path approach and the barycenter approach will be discussed in Sec. X. After numerous technical discussions with the author, Hooker abandoned the barycenter approach in favor of the direct path approach and developed another analysis.[17] We agreed that a great deal of unnecessary calculations can be eliminated.

In the direct path approach, we try to predetermine analytically the contribution from the motion of each body to the certain slots of the coefficient matrices of the overall dynamic equations in the most efficient way. Three approaches are suggested in this analysis, namely; the Lagrangian-Newtonian approach, the complete Newtonian approach, and the complete Lagrangian approach. In the first two approaches, the dynamic equations of each body are first derived from the Lagrangian approach or the Newtonian approach. Since bodies are connected to each other by gimbal hinges, the position vector, the angular velocity vector, and their time derivatives for each body are related to those of the other bodies through kinematic relations.

It is necessary to transform the variables in the dynamic equations of all the bodies to a common set of variables of a specific body in the topological tree system. Evidently, the main body, which carries the payload, is the most logical choice. The direct path is a vectorial path oriented from the main body directly toward each body in the system. The direct position vectors, which lead from the hinge points to the instantaneous mass centers of the other bodies within the same direct path, are widely used throughout the analysis. They serve as the grouping devices of the position vectors embedded in the bodies along the direct path, and transform the influence of the motion of one body to the others. The angular velocity and acceleration vectors of each body are also transformed into those of the main body and the reference coordinates of the system. The relative angular velocity and and acceleration vectors between adjacent bodies along the direct path are also induced into the equations. The incidence matrix used in the analysis can not only uniquely define the specific topological tree configuration, but also systematically govern the transformation of variables of each body. The dynamic equations of each body are rearrranged in such a way that the elimination of the interacting forces and torques at all the hinge points will be completed on one single summation step. This part of the derivation is considered to be Newtonian in nature.

In the complete Lagrangian approach, we first derive the kinetic energy of each body in its own body coordinates, and then transform the variables before obtaining the total kinetic energy of the system. The strain energy and the rate of change in damping energy dissipation in the flexible terminal bodies and at all the gimbal hinges are obtained in a rather simple manner. The overall dynamic equations are directly obtained from LaGrange's equations.

II. System Definition for the Topological Tree

Figure 1 shows a flexible multibody spacecraft with topological tree configuration. Bodies are branching out from the main body into different levels. The main body is named as body 1; the next N_1 level-1 bodies are named as body 2, body 3,...body (N_1+1); and then the next N_2 level-2 bodies are named as body (N_1+2), body (N_1+3),...body (N_1+N_2+1) and so forth. Each body has its own body coordinates, $x^j_\beta\ (j=1,2,...N)$ $(\beta=1,2,3)$. There are also the arbitrarily chosen spacecraft reference coordinates x^r_γ $(\gamma=1,2,3)$, and the Earth coordinates, x^e_ξ $(\xi=1,2,3)$.

III. Dynamic Equations of a Flexible Terminal Body

The dynamic equations of a flexible terminal body may be derived from either the Lagrangian approach or the Newtonian approach.

A. Lagrangian Approach

The position vector of a mass element dm^j in body j at the deformed state with respect to the origin of the Earth coordinates, point O_e, as shown in Fig. 2, is

$$\boldsymbol{\rho}^j = \boldsymbol{R}^r + \boldsymbol{R}^j + \boldsymbol{r}^j + \boldsymbol{u}^j \tag{1}$$

where

$$\boldsymbol{u}^j = \sum_{l=1}^{n_j} q^j_l \boldsymbol{\phi}^j_l \tag{2}$$

The velocity vector of the mass element dm^j is

$$\boldsymbol{v}^j = \dot{\boldsymbol{R}}^r + \dot{\boldsymbol{R}}^j + \boldsymbol{\omega}^r \times \boldsymbol{R}^j + \boldsymbol{\omega}^j \times \boldsymbol{r}^j + \sum_{l=1}^{n_j} (\dot{q}^j_l \boldsymbol{\phi}^j_l + q^j_l \boldsymbol{\omega}^j \times \boldsymbol{\phi}^j_l) \tag{3}$$

The kinetic energy of body j takes the following form

$$\mathcal{K}^j = \tfrac{1}{2}\int \boldsymbol{v}^j . \boldsymbol{v}^j dm^j \tag{4}$$

Substituting Eq. (3) into Eq. (4), we have

$$\mathcal{K}^j = \tfrac{1}{2} m^j [\dot{\boldsymbol{R}}^r + \dot{\boldsymbol{R}}^j - \boldsymbol{R}^j \times \boldsymbol{\omega}^r \,\vert\, \boldsymbol{\omega}^j \,\vert\, \dot{q}^j_k]$$

$$\left[\begin{array}{c|c|c} \delta\cdot & -d^{j^*}\times & \Phi_l^j \\ \hline d^{j^*}\times & B^j\cdot & W_l^j \\ \hline \Phi_k^j\cdot & W_k^j\cdot & M_{kl}^j \end{array}\right] \cdot \left[\begin{array}{c} \dot{R}^r+\dot{R}^j+\omega^r\times R^j \\ \hline \omega^j \\ \hline \dot{q}_l^j \end{array}\right] \tag{5}$$

where the time-dependent mass properties of body j are

$$m^j \quad = \mathrm{d}m^j \tag{6a}$$

$$d^j \quad = (1/m^j)\int r^j \ m^j \tag{6b}$$

$$\Phi_l^j \quad = (1/m^j)\int \phi_l^j \mathrm{d}m^j \tag{6c}$$

$$Y_l^j \quad = (1/m^j)\int r^j \times \phi_l^j \mathrm{d}m^j \tag{6d}$$

$$Z_{kl}^j \quad = (1/m^j)\int \phi_k^j \times \phi_l^j \mathrm{d}m^j \tag{6e}$$

$$M_{kl}^j \quad = (1/m^j)\int \phi_k^j \cdot \phi_l^j \mathrm{d}m^j \tag{6f}$$

$$I^j \quad = (1/m^j)\int [\delta^j(r^j\cdot r^j) - r^j r^j]\mathrm{d}m^j \tag{6g}$$

$$N_l^j \quad = (1/m^j)\int [\delta^j(r^j\cdot\phi_l^j) - r^j\phi_l^j]\mathrm{d}m^j \tag{6h}$$

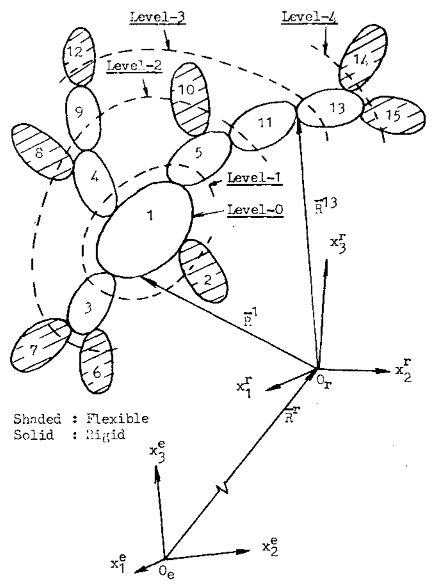

Fig. 1 Flexible multibody spacecraft system with topological tree configuration.

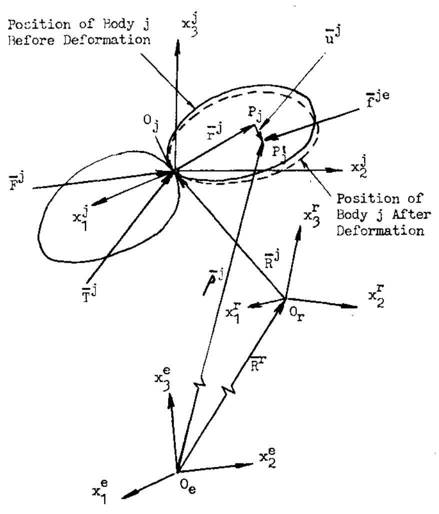

Fig. 2 Positions of a flexible terminal body before and after elastic deformation.

$$N_k^{j^*} \quad = (1/m^j)\int [\delta^j(\phi_k^j\cdot r^j) - \phi_k^j]\mathrm{d}m^j \tag{6i}$$

$$E_{kl}^j \quad = (1/m^j\int [\delta^j(\phi_k^j\cdot\phi_l^j) - \phi_k^j\phi_l^j]\mathrm{d}m^j \tag{6j}$$

$$(k=1,2,\ldots n_j)$$

and the time-dependent mass properties of body j are

$$d^{j^*} = d^j + \sum_{l=1}^{n_j} q_l^j \Phi_l^j \tag{7a}$$

$$B^j = I^j + \sum_{l=1}^{n_j} q_l^j (N_l^j + N_l^{j^*}) + \sum_{k=1}^{n_j}\sum_{l=1}^{n_j} q_k^j q_l^j E_{kl}^j \tag{7b}$$

$$W_k^j = Y_k^j + \sum_{l=1}^{n_j} q_l^j Z_{lk}^j \qquad (k=1,2,\ldots n_j) \tag{7c}$$

The strain of energy of body j is

$$\mathcal{S}^j = \tfrac{1}{2} m^j \sum_{k=1}^{n_j}\sum_{l=1}^{n_j} q_k^j q_l^j K_{kl}^j \tag{8}$$

and the rate of change in damping energy dissipation of body j is

$$\mathfrak{D}^j = \tfrac{1}{2} m^j \sum_{k=1}^{n_j}\sum_{l=1}^{n_j} \dot{q}_k^j \dot{q}_l^j V_{kl}^j \tag{9}$$

The angular velocity vector ω^j of body j is related to the angular velocity vector ω^r of the reference coordinates as

$$\omega^j = \omega^r + G^{jr}\cdot\dot{\theta}^{jr} \tag{10}$$

where

$$\dot{\theta}^{jr} = \dot{\theta}_\eta^{jr} e_\eta^{jr} \qquad G^{jr} = e_\beta^j G_{\beta\zeta}^{jr} e_\zeta^{jr^*} \tag{11}$$

e_η^{jr} $(\eta=1,2,3)$ are the base vectors along the Euler rotational axes x_η^{jr} between the reference coordinates x_γ^r and the body j coordinates x_β^j; and $e_\delta^{jr^*}$ $(\delta=1,2,3)$ are the reciprocal base vectors. They are related by

$$e_1^{jr^*} = (e_2^{jr}\times e_3^{jr})/(e_1^{jr})/(e_1^{jr}\times e_2^{jr}\cdot e_3^{jr}) \tag{12a}$$

$$e_2^{jr} = (e_3^{jr}\times e_1^{jr})/(e_1^{jr}\times e_2^{jr}\cdot e_3^{jr}) \tag{12b}$$

$$e_3^{jr^*} = (e_1^{jr}\times e_2^{jr})/(e_1^{jr}\times e_2^{jr}\cdot e_3^{jr}) \tag{12c}$$

and

$$e_\zeta^{jr^*}\cdot e_\eta^{jr} = \xi_{\zeta\eta} = \text{Kronecker delta} \tag{13}$$

The components of the Euler transformation tensor G^{jr} are

$$G_{\beta\zeta}^{jr} = \begin{bmatrix} C_2C_3 & S_3 & 0 \\ -C_2S_3 & C_3 & 0 \\ S_2 & 0 & 1 \end{bmatrix}^{jr} \tag{14}$$

where

$$C_\eta^{jr} = \cos\theta_\eta^{jr} \quad S_\eta^{jr} = \sin\theta_\eta^{jr} \quad (\eta=1,2,3) \tag{15}$$

The unit vectors, $e_\beta^j(\beta=1,2,3)$ and $e_\gamma^r(\gamma=1,2,3)$ are related by

$$e_\beta^j = A_{\beta\gamma}^{jr} e_\gamma^r \tag{16}$$

where $A_{\beta\gamma}^{jr}$ is called the coordinate transformation matrix.

$$A^{jr}_{\beta\gamma} = \begin{bmatrix} C_2C_3 & (C_1S_3+S_1S_2C_3) & (S_1S_3-C_1S_2C_3) \\ -C_2S_3 & (C_1C_3-S_1S_2S_3) & (S_1C_3+C_1S_2S_3) \\ S_2 & -S_1C_2 & C_1C_2 \end{bmatrix}^{jr} \tag{17}$$

Lagrange's equations of motion of body j are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{K}^j}{\partial \dot{R}^j_\mu}\right] - \frac{\partial \mathcal{K}^j}{\partial R^j_\mu} + \frac{\partial \mathcal{S}^j}{\partial R^j_\mu} + \frac{\partial \mathcal{D}^j}{\partial \dot{R}^j_\mu} = \boldsymbol{e}^r_\mu \cdot (\boldsymbol{F}^j + \boldsymbol{F}^{je}) \tag{18a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{K}^j}{\partial \dot{\theta}^{jr}_\eta}\right] - \frac{\partial \mathcal{K}^j}{\partial \theta^{jr}_\eta} + \frac{\partial \mathcal{S}^j}{\partial \theta^{jr}_\eta} + \frac{\partial \mathcal{D}^j}{\partial \dot{\theta}^{jr}_\eta} = \boldsymbol{e}^{jr}_\eta \cdot \boldsymbol{G}^{jr(T)} \cdot (\boldsymbol{T}^j + \boldsymbol{T}^{je}) \tag{18b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \mathcal{K}^j}{\partial \dot{q}^j_k}\right] - \frac{\partial \mathcal{K}^j}{\partial q^j_k} + \frac{\partial \mathcal{S}^j}{\partial q^j_k} + \frac{\partial \mathcal{D}^j}{\partial \dot{q}^j_k} = Q^{je}_k \tag{18c}$$

$$(\mu = 1,2,3;\ \eta = 1,2,3;\ k = 1,2,\ldots n_j)$$

where the environmental disturbances on body j are

$$\boldsymbol{F}^{je} = \int \boldsymbol{f}^{je}\mathrm{d}m^j \tag{19a}$$

$$\boldsymbol{T}^{je} = \int (\boldsymbol{r}^j + \boldsymbol{u}^j) \times \boldsymbol{f}^{je}\mathrm{d}m^j \tag{19b}$$

$$Q^{je}_k = \int \boldsymbol{\phi}^j_k \cdot \boldsymbol{f}^{je}\mathrm{d}m^j \quad (k = 1,2,\ldots n_j) \tag{19c}$$

Substituting Eqs. (5), (8), and (9) into Eqs. (18), we obtain the dynamic equations of the flexible terminal body, body j

$$m^j \begin{bmatrix} \boldsymbol{\delta}\cdot & -\boldsymbol{d}^{j^*}\times & \boldsymbol{\Phi}^j_l \\ \boldsymbol{d}^{j^*}\times & \boldsymbol{B}^j\cdot & \boldsymbol{W}^j_l \\ \boldsymbol{\Phi}^j_k\cdot & \boldsymbol{W}^j_k\cdot & M^j_{kl} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{R}}^r + \overset{\circ\circ}{\boldsymbol{R}}{}^j \\ \dot{\boldsymbol{\omega}}^j \\ \ddot{q}^j_l \end{Bmatrix} + m^j \begin{Bmatrix} \boldsymbol{\omega}^j \times (\boldsymbol{\omega}^j \times \boldsymbol{d}^{j^*}) \\ \boldsymbol{\omega}^j \times \boldsymbol{B}^j \cdot \boldsymbol{\omega}^j \\ -\boldsymbol{\omega}^j \cdot \boldsymbol{C}^j_k \cdot \boldsymbol{\omega}^j \end{Bmatrix} + 2m^j \begin{Bmatrix} \boldsymbol{\omega}^j \times \boldsymbol{\Phi}^j_l \\ \boldsymbol{C}^j_l \cdot \boldsymbol{\omega}^j \\ -\boldsymbol{Z}^j_{kl} \cdot \boldsymbol{\omega}^j \end{Bmatrix} \dot{q}^j_l$$

$$+ m^j \begin{Bmatrix} 0 \\ 0 \\ K^j_{kl}q^j_l + V^j_{kl}\dot{q}^j_l \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}^{je} \\ \boldsymbol{T}^{je} \\ Q^{je}_k \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{F}^j \\ \boldsymbol{T}^j \\ 0 \end{Bmatrix} \qquad \begin{matrix} (20a) \\ (20b) \\ (20c) \end{matrix}$$

$(k = 1,2,\ldots n_j;$ sum on $l = 1,2,\ldots n_j)$

where we have the additional time-dependent mass properties of body j.

$$\boldsymbol{C}^j_l = \boldsymbol{N}^{j^*}_l + \sum_{k=1}^{n_j} q^j_k \boldsymbol{E}^j_{kl} \qquad (l = 1,2,\ldots n_j) \tag{21}$$

and

$$\overset{\circ\circ}{\boldsymbol{R}}{}^j = \ddot{\boldsymbol{R}}^j + 2\boldsymbol{\omega}^r \times \dot{\boldsymbol{R}}^j + \dot{\boldsymbol{\omega}}^r \times \boldsymbol{R}^j + \boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \boldsymbol{R}^j) \tag{22}$$

B. Newtonian Approach

The Newtonian approach of deriving the dynamic equations of the flexible terminal body is based on the balancing of forces, torques, energy, and work of the body. It requires the acceleration vector of the mass element $\mathrm{d}m^j$.

$$\boldsymbol{a}^j = \ddot{\boldsymbol{R}}^r + \overset{\circ\circ}{\boldsymbol{R}}{}^j + \dot{\boldsymbol{\omega}}^j \times \boldsymbol{r}^j + \boldsymbol{\omega}^j \times (\boldsymbol{\omega}^j \times \boldsymbol{r}^j)$$
$$+ \sum_{l=1}^{n_j} \{ \ddot{q}^j_l \boldsymbol{\phi}^j_l + 2\dot{q}^j_l \boldsymbol{\omega}^j \times \boldsymbol{\phi}^j_l + q^j_l [\dot{\boldsymbol{\omega}}^j \times \boldsymbol{\phi}^j_l + \boldsymbol{\omega}^j \times (\boldsymbol{\omega}^j \times \boldsymbol{\phi}^j_l)] \} \tag{23}$$

The dynamic equations of motion of the flexible terminal body are

$$\int \boldsymbol{a}^j \mathrm{d}m^j = \boldsymbol{F}^{je} + \boldsymbol{F}^j \tag{24a}$$

$$\int (\boldsymbol{r}^j + \boldsymbol{u}^j) \times \boldsymbol{a}^j \mathrm{d}m^j = \boldsymbol{T}^{je} + \boldsymbol{T}^j \tag{24b}$$

$$\int \boldsymbol{\phi}^j_k \cdot \boldsymbol{a}^j \mathrm{d}m^j + m^j \sum_{l=1}^{n_j} [q^j_l K^j_{kl} + \dot{q}^j_l V^j_{kl}] = Q^{je}_k \tag{24c}$$

$$(k = 1,2,\ldots n_j)$$

Substituting Eq. (23) into Eqs. (24) will yield the same equations as Eq. 20.

IV. Dynamic Equations of a Rigid Interconnected Body

When body j is a rigid interconnected body, the dynamic equations of motion will be reduced into the following simple form

$$m^j \begin{bmatrix} \boldsymbol{\delta}\cdot & -\boldsymbol{d}^j\times \\ \boldsymbol{d}^j\times & \boldsymbol{I}^j\cdot \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{R}}^r + \overset{\circ\circ}{\boldsymbol{R}}{}^j \\ \dot{\boldsymbol{\omega}}^j \end{Bmatrix} + m^j \begin{Bmatrix} \boldsymbol{\omega}^j \times (\boldsymbol{\omega}^j \times \boldsymbol{d}^j) \\ \boldsymbol{\omega}^j \times \boldsymbol{I}^j \cdot \boldsymbol{\omega}^j \end{Bmatrix}$$

$$= \begin{Bmatrix} \boldsymbol{F}^{je} \\ \boldsymbol{T}^{je} \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{F}^j - \sum_m^{(j)} \boldsymbol{F}^m \\ \boldsymbol{T}^j - \sum_m^{(j)} (\boldsymbol{T}^m + \boldsymbol{h}^{jm} \times \boldsymbol{F}^m) \end{Bmatrix} \tag{25}$$

where the special summation $\Sigma^{(j)}_m$ only sums up the adjacent branch bodies of body j.

V. Direct Path Method for Transforming Variables

The variables in the dynamic equations of a typical body (flexible terminal or rigid interconnected) $\boldsymbol{R}^j$ and $\boldsymbol{\omega}^j$ and their time derivatives are related to those of the other bodies through the kinematic relations. The variables q^j_l $(l = 1,2,\ldots n_j)$ and their time derivatives are independent of those of the other bodies. To obtain the overall dynamic equations of the complete system, one must transform the set of the interrelated variables into a common set. Evidently, the main body, body 1, is the most logical choice.

The concept of the direct path is hereby introduced (Fig. 3). To go from body 1 to an arbitrary body, body j, in the toplоogical tree system, there always exists a body path which is the most direct path. This body path is called the direct path 1-j. Let body v and body p by the typical bodies in the direct path 1-j. We also have the direct path v-j and the direct path p-j.

We now define the direct position vector $\boldsymbol{g}^{v(j)}$ as the vector from the origin of the boyd v coordinates, point O_v, to the instantaneous mass center of body j, point Q'_j. Next we in-

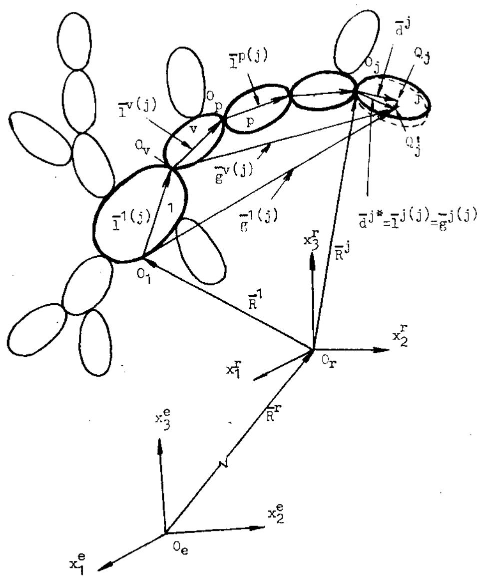

Fig. 3 Direct paths of body j and direct position vectors.

troduce the "incidence matrix," ϵ^{vj} $(v,j=1,2,\ldots N)$. The values of ϵ^{vj} are either 1 or 0. They govern the appearance of body v in the direct path 1-j for the specific topological tree. The incidence matrix for the topological tree in Fig. 1 is shown in Table 1.

The direct position vector is written as

$$g^{v(j)}=\epsilon^{vj}\sum_{p=v}^{j}\epsilon^{pj}l^{p(j)} \tag{26}$$

where $l^{p(j)}=l_\beta^{p(j)}e_\beta^p=$ position vector $\overline{O_pO_s}$, where body s is the adjacent branch body of body p; and both body p and body s are in the direct path v-j $(p\neq j)$ and also $l^{j(j)}=d^{j^*}=g^{j(j)}$.

$$R^j=R^1+\sum_{p=1}^{j}\epsilon^{pj}l^{p(j)}-l^{j(j)} \tag{27a}$$

$$\dot{R}^j+\omega^r\times R^j=\dot{R}^1+\omega^r\times R^1+\sum_{p=1}^{j}\epsilon^{pj}[\omega^p\times l^{p(j)}]-\omega^j\times l^{j(j)} \tag{27b}$$

$$\overset{\circ\circ}{R}{}^j=\overset{\circ\circ}{R}{}^1+\sum_{p=1}^{j}\epsilon^{pj}[\dot{\omega}^p\times l^{p(j)}+\omega^p(\omega^p\times l^{p(j)})]$$
$$-\dot{\omega}^j\times l^{j(j)}-\omega^j\times(\omega^j\times l^{j(j)}) \tag{27c}$$

Since

$$\omega^p=\omega^r+\sum_{v=1}^{p}\epsilon^{vp}[G^v\cdot\dot{\theta}^v] \tag{28a}$$

$$\dot{\omega}^p=\dot{\omega}^r+\sum_{v=1}^{p}\epsilon^{vp}[\dot{G}^v\cdot\ddot{\theta}^v+(\dot{G}^v+\omega^v\times G^v)\cdot\dot{\theta}^v] \tag{28b}$$
$$=\dot{\omega}^r+\sum_{v=1}^{p}\epsilon^{vp}[G^v\cdot\dot{\theta}^v]^\circ$$

where

$$\dot{\theta}^v=\dot{\theta}_\lambda^v e_\lambda^{vg}\qquad G^v=e_\beta^v G_{\beta\mu}^v e_\mu^{vg^*} \tag{29}$$

e_λ^{vg} ($\lambda=1,2,3$ are the base vectors along the gimbal rotational axes x_λ^{vg} between body v and its limb body; and $e_\mu^{vg^*}$ $(\mu=1,2,3)$ are the reciprocal base vectors. Their relations are similar to those in Eqs. (12) and (13).

Table 1 Incidence matrix for the topological tree in Fig. 1[a]

v \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1				1	1								
3			1			1	1								
4				1				1	1			1			
5					1					1	1		1	1	1
6						1									
7							1								
8								1							
9									1						
10										1					
11											1		1	1	1
12												1			
13													1	1	1
14														1	
15															1

[a]The empty slots are zeros.

Combining Eqs. (28), (29), and (26) yields the following equation

$$\overset{\circ\circ}{R}{}^j=\overset{\circ\circ}{R}{}^1-(g^{1(j)}-g^{j(j)})\times\dot{\omega}^r$$
$$-\sum_{v=1}^{j}\epsilon^{vj}[g^{v(j)}-g^{j(j)}]\times[G^v\cdot\dot{\theta}^v]^\circ$$
$$+\sum_{v=1}^{j}\epsilon^{vj}[\omega^v\times(\omega^v\times l^{v(j)})]-\omega^j(\omega^j\times l^{j(j)}) \tag{30}$$

VI. Contribution from Single-Body Motion to Overall Dynamic Equations

Before determining the contribution from the single-body motion to the overall dynamic equations of the system, let us reexamine the physical meaning of Eqs. (20a,b,c): Equation (20a)—force equation of body j; Equation (20b)—torque equation of body j about point O_j; and Equation (20c)—independent vibration equations of body j. The overall dynamic equations of the system will take the following form: 1) the overall force equation of the system; 2) the torque equations about all the hinge points, O_u $(u=1,2,\ldots N)$; and 3) the independent vibration equation of all the flexible terminal bodies.

The motion of a single body will definitely influence the motion of the whole system. In this direct path method, we have found that we do not really need all the interrelations among bodies. The motion of body j will only contribute to the overall force equation and the torque equations about the hinge points which belong to the *direct path 1-j*. We may rearrange Eqs. (20a,b,c) in the following manner

$$\left\{\begin{array}{c}\text{Eq. (20a)}\\ \hline \epsilon^{uj}[\text{Eq. (20b)}+(g^{u(j)}-g^{j(j)})\times\text{Eq. (20a)}]\\ \hline \epsilon^{sj}[\text{Eq. (20c)}]\end{array}\right\}$$

$(u=1,2,\ldots N;\ s=$ flexible terminal bodies)

Using the previous rearrangement and the transformation equations (28) and (30), the contribution from the motion of body j to the overall dynamic equations of the system will be

$$m^j\left[\begin{array}{c|c|c}\delta\cdot & -g^{v(j)}\epsilon^{vj}\times & \Phi_l^j\epsilon^{lj}\\ \hline \epsilon^{uj}g^{u(j)}\times & \epsilon^{uj}\mathcal{g}^{uv(j)}\epsilon^{vj} & \epsilon^{uj}\omega_l^{u(j)}\epsilon^{lj}\\ \hline \epsilon^{sj}\Phi_k^j\cdot & \epsilon^{sj}\omega_k^{v(j)}\epsilon^{vj} & \epsilon^{sj}M_{kl}^j\epsilon^{lj}\end{array}\right]\left\{\begin{array}{c}\overset{\circ\circ}{R}{}^r+\overset{\circ\circ}{R}{}^1\\ \hline [G^v\cdot\dot{\theta}^v]^\circ\\ \hline \ddot{q}_l^l\end{array}\right\}$$

$$
+m^j \left\{\begin{array}{c} -\boldsymbol{g}^{l(j)}\times \\ \hline \epsilon^{uj}\mathcal{G}^{ul(j)}\boldsymbol{\cdot} \\ \hline \epsilon^{sj}\boldsymbol{\omega}_k^{l(j)}\boldsymbol{\cdot} \end{array}\right\} \dot{\boldsymbol{\omega}}^r + m^j \left\{\begin{array}{c} 0 \\ \hline \epsilon^{uj}\boldsymbol{\omega}^j\times \boldsymbol{B}^{j\Delta}\boldsymbol{\cdot}\boldsymbol{\omega}^j \\ \hline -\epsilon^{sj}\boldsymbol{\omega}^j\boldsymbol{\cdot}\boldsymbol{C}_k^{j\Delta}\boldsymbol{\cdot}\boldsymbol{\omega}^j \end{array}\right\}
$$

$$
+m^j \left\{\begin{array}{c} \boldsymbol{\delta}\boldsymbol{\cdot} \\ \hline \epsilon^{uj}\boldsymbol{g}^{u(j)}\times \\ \hline \epsilon^{sj}\boldsymbol{\Phi}_k^j\boldsymbol{\cdot} \end{array}\right\} \epsilon^{uj}[\boldsymbol{\omega}^v\times(\boldsymbol{\omega}^v\times\boldsymbol{l}^{v(j)})]
$$

$$
+2m^j \left\{\begin{array}{c} \boldsymbol{\omega}^j\times\boldsymbol{\Phi}_l^j \\ \hline \epsilon^{uj}\mathcal{G}_l^{u(j)}\boldsymbol{\cdot}\boldsymbol{\omega}^j \\ \hline -\epsilon^{sj}\boldsymbol{Z}_{kl}^j\boldsymbol{\cdot}\boldsymbol{\omega}^j \end{array}\right\} \epsilon^{tj}\dot{q}_l^t + m^j \left\{\begin{array}{c} 0 \\ \hline 0 \\ \hline \epsilon^{sj}(K_{kl}^j q_l^j + V_{kl}^j \dot{q}_l^j) \end{array}\right\}
$$

$$
= \left\{\begin{array}{c} \boldsymbol{F}^{je} \\ \hline \epsilon^{uj}\boldsymbol{T}^{u(je)} \\ \hline \epsilon^{sj}Q_k^{je} \end{array}\right\} + \left\{\begin{array}{c} \boldsymbol{F}^j \\ \hline \epsilon^{uj}[\boldsymbol{T}^j + (\boldsymbol{g}^{u(j)}-\boldsymbol{g}^{j(j)})\times\boldsymbol{F}^j] \\ \hline 0 \end{array}\right\} \begin{array}{c}(31a)\\(31b)\\(31c)\end{array}
$$

$u=1,2,\ldots N$; $s=$ flexible terminal bodies; $k=1,2,\ldots n_j$; sum on $v=1,2,\ldots N$; $t=$ flexible terminal bodies; and $l=1,2,\ldots n_j$.

where

$$\mathcal{G}^{uv(j)} = \boldsymbol{B}^{j\Delta} - \boldsymbol{g}^{u(j)}\times(\boldsymbol{g}^{v(j)}\times \tag{32a}$$

$$\mathcal{W}_l^{u(j)} = \boldsymbol{W}_l^{j\Delta} + \boldsymbol{g}^{u(j)}\times\boldsymbol{\Phi}_l^j \tag{32b}$$

$$\mathcal{W}_k^{v(j)} = \boldsymbol{W}_k^{j\Delta} - \boldsymbol{\Phi}_k^j\times\boldsymbol{g}^{v(j)} \tag{32c}$$

$$\mathcal{G}_l^{u(j)} = \boldsymbol{C}_l^j - (\boldsymbol{g}^{u(j)} - \boldsymbol{g}^{j(j)})\times(\boldsymbol{\Phi}_l^j\times \tag{32d}$$

$$
m\left[\begin{array}{c|c|c} \boldsymbol{\delta}\boldsymbol{\cdot} & -\boldsymbol{D}^v\times\boldsymbol{G}^v\boldsymbol{\cdot} & \boldsymbol{\Psi}_l^t \\ \hline \boldsymbol{G}^{u(T)}\times\boldsymbol{D}^u\boldsymbol{\cdot} & \boldsymbol{G}^{u(T)}\boldsymbol{\cdot}\boldsymbol{J}^{uv}\boldsymbol{\cdot}\boldsymbol{G}^v\boldsymbol{\cdot} & \boldsymbol{G}^{u(T)}\boldsymbol{\cdot}\boldsymbol{H}_l^{ut} \\ \hline \boldsymbol{\Psi}_k^s & \boldsymbol{H}_k^{sv*}\boldsymbol{\cdot}\boldsymbol{G}^v\boldsymbol{\cdot} & S_{kl}^{st} \end{array}\right] \left\{\begin{array}{c} \ddot{\boldsymbol{R}}^l \\ \hline \ddot{\boldsymbol{\theta}}^v \\ \hline \ddot{q}_l^t \end{array}\right\} = -m\left[\begin{array}{c|c} \boldsymbol{\delta}\boldsymbol{\cdot} & -\boldsymbol{D}^v\times \\ \hline \boldsymbol{G}^{u(T)}\times\boldsymbol{D}^u\boldsymbol{\cdot} & \boldsymbol{G}^{u(T)}\boldsymbol{\cdot}\boldsymbol{J}^{uv}\boldsymbol{\cdot} \\ \hline \boldsymbol{\Psi}_k^s\boldsymbol{\cdot} & \boldsymbol{H}_k^{sv*}\boldsymbol{\cdot} \end{array}\right]
$$

$$
\left\{\begin{array}{c} \ddot{\boldsymbol{R}}^r + 2\boldsymbol{\omega}^r\times\dot{\boldsymbol{R}}^l \\ +\dot{\boldsymbol{\omega}}^r\times\boldsymbol{R}^l + \boldsymbol{\omega}^r\times(\boldsymbol{\omega}^r\times\boldsymbol{R}^l) \\ \hline (\dot{\boldsymbol{G}}^v + \boldsymbol{\omega}^v\times\boldsymbol{G}^v)\boldsymbol{\cdot}\dot{\boldsymbol{\theta}}^v \end{array}\right\} - m\left\{\begin{array}{c} -\boldsymbol{D}^l\times \\ \hline \boldsymbol{G}^{u(T)}\boldsymbol{\cdot}\boldsymbol{J}^{ul}\boldsymbol{\cdot} \\ \hline \boldsymbol{H}_k^{sl*}\boldsymbol{\cdot} \end{array}\right\}\dot{\boldsymbol{\omega}}^r - m\left\{\begin{array}{c} \boldsymbol{L} \\ \hline \boldsymbol{G}^{u(T)}\boldsymbol{\cdot}\boldsymbol{P}^u \\ \hline X_k^s \end{array}\right\} + \left\{\begin{array}{c} 0 \\ \hline \boldsymbol{T}^{ug} \\ \hline 0 \end{array}\right\} + \left\{\begin{array}{c} \boldsymbol{F}^{(e)} \\ \hline \boldsymbol{G}^{u(T)}\boldsymbol{\cdot}\boldsymbol{T}^{u(e)} \\ \hline Q_k^{s(e)} \end{array}\right\} \begin{array}{c}(37a)\\(37b)\\(37c)\end{array}
$$

$$\boldsymbol{B}^{j\Delta} = \boldsymbol{B}^j + \boldsymbol{d}^{j*}\times(\boldsymbol{d}^{j*}\times \tag{32e}$$

$$\boldsymbol{W}_k^{j\Delta} = \boldsymbol{W}_k^j + \boldsymbol{\Phi}_k^j\times\boldsymbol{d}^{j*} \tag{32f}$$

$$\boldsymbol{C}_k^{j\Delta} = \boldsymbol{C}_k^j + \boldsymbol{\Phi}_k^j\times(\boldsymbol{d}^{j*}\times \tag{32g}$$

$$(u,v=1,2,\ldots N;\ k=1,2,\ldots n_j)$$

and

$$\boldsymbol{T}^{u(je)} = \boldsymbol{T}^{je} + (\boldsymbol{g}^{u(j)} - \boldsymbol{g}^{j(j)})\times\boldsymbol{F}^{je} \tag{33}$$

$$(u=1,2,\ldots N)$$

When body j is a rigid interconnected body, there will be no vibration equations, and all the terms associated with q_l^j and $\dot{q}_l^j$ will vanish. The interacting forces and torques will be

$$
\left\{\begin{array}{c} \boldsymbol{F}^j - \sum_m^{(j)}\boldsymbol{F}^m \\ \hline \epsilon^{uj}\{\boldsymbol{T}^j + (\boldsymbol{g}^{u(j)} - \boldsymbol{g}^{j(j)})\times\boldsymbol{F}^j \\ -\sum_m^{(j)}[\boldsymbol{T}^m + (\boldsymbol{g}^{u(j)} - \boldsymbol{g}^{j(j)} + \boldsymbol{h}^{jm})\times\boldsymbol{F}^m]\} \end{array}\right\}
$$

VII. Overall Dynamic Equations of System

The overall dynamic equations of the multibody system can be obtained by the following simple summation.

$$\sum_{j=l}^{N}\left\{\boldsymbol{F}^j - \sum_m^{(j)}\boldsymbol{F}^m\right\} = \boldsymbol{F}^l = 0 \tag{34}$$

$$\sum_{j=u}^{N}\epsilon^{uj}\Big\{\boldsymbol{T}^j + (\boldsymbol{g}^{u(j)} - \boldsymbol{g}^{j(j)})\times\boldsymbol{F}^j - \sum_m^{(j)}[\boldsymbol{T}^m + (\boldsymbol{g}^{u(j)} - \boldsymbol{g}^{j(j)} + \boldsymbol{h}^{jm})\times\boldsymbol{F}^m]\Big\} = \boldsymbol{T}^u \tag{35}$$

$\boldsymbol{T}^u$ is the interacting torque vector at point O_u and its components T_β^u $(\beta=1,2,3)$ are associated with the orthogonal unit vectors $\boldsymbol{e}_\beta^u$ in the body u coordinates. It can be transformed into the gimbal torque $\boldsymbol{T}^{ug}$ by premultiplying Eq. (35) with the transpose gimbal transformation tensor $\boldsymbol{G}^{u(T)} = \boldsymbol{e}_\mu^{ug*}G_{\mu\beta}^{u(T)}\boldsymbol{e}_\beta^u$. Since the components of the gimbal torque $\boldsymbol{T}^{ug}$, i.e., T_λ^{ug} $(\lambda=1,2,3)$, are associated with the nonorthogonal base vectors $\boldsymbol{e}_\lambda^{ug}$ $(\lambda=1,2,3)$ along the gimbal rotational axes x_λ^{ug} $(\lambda=1,2,3)$, we have the following relation

$$
\begin{aligned}
\boldsymbol{T}^{ug} &= \boldsymbol{e}_\lambda^{ug}T_\lambda^{ug} = \boldsymbol{G}^{u(T)}\cdot\boldsymbol{T}^u = \boldsymbol{e}_\mu^{ug*}G_{\mu\beta}^{u(T)}T_\beta^u \\
&= \boldsymbol{e}_\lambda^{ug}G_{\lambda\zeta}^{u(-1)}G_{\zeta\mu}^{u(T)(-1)}G_{\mu\lambda}^{u(T)}T_\beta^u \\
&= \boldsymbol{e}_\lambda^{ug}G_{\lambda\beta}^{u(-1)}T_\beta^u = \boldsymbol{e}_\lambda^{ug}(-K_\lambda^{ug}\theta_\lambda^u - V_\lambda^{ug}\dot{\theta}_\lambda^u)
\end{aligned} \tag{36}
$$

The interacting forces and torques are now completely eliminated, because θ_λ^u and $\dot{\theta}_\lambda^u(\lambda=1,2,3)$ are part of the state variables in the overall dynamic equations which will take the following form

where

$$m = \sum_{j=1}^{N} m^j \tag{38a}$$

$$\mu^j = m^j/m \tag{38b}$$

$$\boldsymbol{D}^v = \sum_{j=v}^{N}\mu^j\epsilon^{vj}\boldsymbol{g}^{v(j)} \tag{38c}$$

$$\boldsymbol{\Psi}_l^t = \mu^t\boldsymbol{\Phi}_l^t \tag{38d}$$

$$\boldsymbol{J}^{uv} = \sum_{j=\max(u,v)}^{N}\mu^j\epsilon^{uj}\mathcal{G}^{uv(j)}\epsilon^{vj} \tag{38e}$$

$$\boldsymbol{H}_l^{ut} = \epsilon^{ut}\mu^t\mathcal{W}_l^{u(t)} \tag{38f}$$

$$\boldsymbol{H}_k^{sv*} = \epsilon^{vs}\mu^s\mathcal{W}_k^{v(s)} \tag{38g}$$

$$S_{kl}^{st} = \epsilon^{st}\mu^t M_{kl}^t \tag{38h}$$

$u,v=1,2,\ldots N$; $s,t=$ flexible terminal bodies; $k=1,2,\ldots n_s$; and $l=1,2,\ldots n_t$.

$$L=\sum_{j=1}^{N}\mu^j\sum_{v=1}^{N}\epsilon^{vj}\omega^v\times(\omega^v\times l^{v(j)})+2\sum_{t}^{(F)}\mu^t\sum_{l=1}^{n_t}\dot q_l^t\omega^t\times\Phi_l^t \tag{39a}$$

$$P^u=\sum_{j=u}^{N}\mu^j\epsilon^{uj}\omega^j\times B^{j\Delta}\cdot\omega^j+\sum_{j=u}^{N}\mu^j\epsilon^{uj}\sum_{v=1}^{N}\epsilon^{vj}g^{u(j)}$$

$$\times[\omega^v\times(\omega^v\times l^{v(j)})]+2\sum_{t}^{(F)}\mu^t\epsilon^{ut}\sum_{l=1}^{n_t}\dot q_l^t\mathcal{J}_l^{u(t)}\cdot\omega^t \tag{39b}$$

$$X_k^s=\mu^s\Big\{-\omega^s\cdot C_k^{s\delta}\cdot\omega^s+\Phi_k^s\cdot\sum_{v=1}^{N}\epsilon^{vs}\omega^v\times(\omega^v\times l^{v(s)})$$

$$-2\sum_{l=1}^{n^s}\dot q_l^s Z_{kl}^s\cdot\omega^s+\sum_{l=1}^{n^s}\Big[q_l^s K_{kl}^s+\dot q_l^s V_{kl}^s\Big]\Big\} \tag{39c}$$

$u=1,2,\ldots N$; $s=$ flexible terminal bodies; and $k=1,2,\ldots n_s$.

$$F^{(e)}=\sum_{j=1}^{N}F^{je}\qquad T^{u(e)}=\sum_{j=u}^{N}\epsilon^{uj}T^{u(je)} \tag{40}$$

VIII. Complete Lagrangian Approach

We now propose an alternative approach, called the complete Lagrangian approach. The first step is to obtain the total kinetic energy of the system.

$$\mathcal{K}=\sum_{j=1}^{N}\mathcal{K}^j \tag{41}$$

Substituting the transformation equations, (27) and (28) into Eqs. (5) and (41), we have

$$\mathcal{K}=\tfrac{1}{2}m[\dot R^r+\mathring{R}^l \mid \omega^r \mid \dot\theta^u\cdot G^{u(T)} \mid \dot q_k^s]$$

$$\cdot\left[\begin{array}{c|c|c|c} \delta\cdot & -D^l\times & -D^v\times & \Psi_l^t \\ \hline D^l\times & J^{ll}\cdot & J^{lv}\cdot & H_l^{lt} \\ \hline D^u\times & J^{ul}\cdot & J^{uv}\cdot & H_l^{ut} \\ \hline \Psi_k^s\cdot & H_k^{sl^*}\cdot & H_k^{sv^*}\cdot & S_{kl}^{st} \end{array}\right]\left\{\begin{array}{c} \dot R^r+\mathring{R}^l \\ \hline \omega^r \\ \hline G^v\cdot\dot\theta^v \\ \hline \dot q_l^t \end{array}\right\} \tag{42}$$

$u,v=1,2,\ldots N$; $s,t=$ flexible terminal bodies; $k=1,2,\ldots n_s$; $l=1,2,\ldots n_t$, where

$$\mathring{R}^l=\dot R^l+\omega^r\times R^l \tag{43}$$

The total strain energy of the system is

$$\mathcal{S}=\tfrac{1}{2}m\sum_{j}^{(F)}\mu^j\sum_{k=1}^{n_j}\sum_{l=1}^{n_j}q_k^jq_l^jK_{kl}^j+\tfrac{1}{2}\sum_{u=2}^{N}\sum_{\sigma=1}^{3}\theta_\sigma^uK_\sigma^{ug}\theta_\sigma^u \tag{44}$$

$$\mathfrak{D}=\tfrac{1}{2}m\sum_{j}^{(F)}\mu^j\sum_{k=1}^{n_j}\sum_{l=1}^{n_j}\dot q_k^j\dot q_l^jV_{kl}^j+\tfrac{1}{2}\sum_{u=2}^{N}\sum_{\sigma=1}^{3}\dot\theta_\sigma^uV_\sigma^{ug}\dot\theta_\sigma^u \tag{45}$$

The Lagrange's equations of the complete system are

$$\frac{d}{dt}\left(\frac{\partial\mathcal{K}}{\partial\dot R_\gamma^l}\right)-\frac{\partial\mathcal{K}}{\partial R_\gamma^l}+\frac{\partial\mathcal{S}}{\partial R_\gamma^l}+\frac{\partial\mathfrak{D}}{\partial\dot R_\gamma^l}=Q_\gamma^R \tag{46a}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{K}}{\partial\dot\theta_\sigma^u}\right)-\frac{\partial\mathcal{K}}{\partial\theta_\sigma^u}+\frac{\partial\mathcal{S}}{\partial\theta_\sigma^u}+\frac{\partial\mathfrak{D}}{\partial\dot\theta_\sigma^u}=Q_\sigma^{\theta u} \tag{46b}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{K}}{\partial\dot q_k^s}\right)-\frac{\partial\mathcal{K}}{\partial q_k^s}+\frac{\partial\mathcal{S}}{\partial q_k^s}+\frac{\partial\mathfrak{D}}{\partial\dot q_k^s}=Q_k^{qs} \tag{46c}$$

$\gamma=1,2,3$; $u=1,2,\ldots N$; $\sigma=1,2,3$; $s=$ flexible terminal bodies; $k=1,2,\ldots n_s$. Since the interacting forces and torques at the hinge points always come in equal and opposite pairs, they will not contribute to any virtual work associated with the virtual displacements. Consequently, they will not appear in the generalized forces of Lagrange's equations. Substituting Eqs. (42), (44), and (45) into Eqs. (46), we shall obtain exactly the same dynamic equations of the multibody system as Eqs. (37).

IX. Limited Degrees of Freedom

When a gimbal rotational axis at a certain hinge point is locked, we simply strike out the torque component equation and the associated angle of rotation about that axis. Projection from the body torque equations to the gimbal torque equations makes the modification for the limited degree of freedom rather simple. [see Eq. (37b)].

X. Discussion and Conclusion

The vector D^u is a position vector from the hinge point O_u to the instantaneous mass center O_u^* of the subsystem of bodies of body u, including itself. The torque equations in Eq. (37b) are taken about the hinge point O_u $(u=1,2,\ldots N)$. We may perform the following shifting process to eliminate the force equation (37a)

$$\left\{\begin{array}{c}\text{Eq.(37b)}-G^{u(T)}\times D^u\cdot\text{Eq.(37a)}\\ \hline \text{Eq.(37c)}-\Psi_k^s\cdot\text{Eq.(37a)}\end{array}\right\}$$

The overall dynamic equations of the system will be

$$m\left[\begin{array}{c|c} G^{u(T)}\cdot[J^{uv}+D^u\times(D^v\times]\cdot G^v & G^{u(T)}\cdot[H_l^{ut}-D^u\times\Psi_l^t] \\ \hline [H_k^{sv^*}+\Psi_k^s\times D^v]\cdot G^v & S_{kl}^{st}-\Psi_k^s\cdot\Psi_l^t \end{array}\right]\cdot\left\{\begin{array}{c}\ddot\theta^v\\ \hline \ddot q_l^t\end{array}\right\}=-m\left\{\begin{array}{c}G^{u(T)}\cdot[J^{uv}+D^u\times(D^v\times]\\ \hline H_k^{sl^*}+\Psi_k^s\times D^v\end{array}\right\}\cdot[\dot G^v+\omega^v\times G^v]\cdot\dot\theta^v$$

$$-m\left\{\begin{array}{c}G^{u(T)}\cdot[J^{ul}+D^u\times(D^l\times]\\ \hline H_k^{sl^*}+\Psi_k^s\times D^l\end{array}\right\}\cdot\dot\omega^{r'}-m\left\{\begin{array}{c}G^{u(T)}\cdot[P^u-D^u\times L]\\ \hline X_k^s-\Psi_k^s\cdot L\end{array}\right\}+\left\{\begin{array}{c}T^{ug}\\ \hline 0\end{array}\right\}+\left\{\begin{array}{c}G^{u(T)}\cdot[T^{u(e)}-D^u\times F^{(e)}]\\ \hline Q_k^{s(e)}-\Psi_k^s\cdot F^{(e)}\end{array}\right\} \tag{47}$$

$u=1,2,\ldots N$; $s=$ flexible terminal bodies; $k=1,2,\ldots n$; sum on $v=1,2,\ldots N$; $t=$ flexible terminal bodies; $l=1,2,\ldots n_t$.

When the direct path method was first developed by the author, Hooker asked two very important questions. The first

Table 2 Incidence matrix of an 8-body system[a]

v \ j	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
3		1						
3			1		1	1		
4				1			1	1
5					1			
6						1		
7							1	1
8								1

[a] The empty slots are zeros

Table 3 Contribution from the body motion to the coefficient matrices in the overall dynamic equations of an 8-body system

1,2,3,4 5,6,7,8	1,2,3,4 5,6,7,8	2	3,5,6	4,7,8	5	6	7,8	8	2	5	6	8
1,2,3,4 5,6,7,8	1,2,3,4 5,6,7,8	2	3,5,6	4,7,8	5	6	7,8	8	2	5	6	8
2	2	2	0	0	0	0	0	0	2	0	0	0
3,5,6	3,5,6	0	3,5,6	0	5	6	0	0	0	5	6	0
4,7,8	4,7,8	0	0	4,7,8	0	0	7,8	8	0	0	0	0
5	5	0	5	0	5	0	0	0	0	5	0	0
6	6	0	6	0	0	6	0	0	0	0	6	0
7,8	7,8	0	0	7,8	0	0	7,8	8	0	0	0	8
8	8	0	0	8	0	0	8	8	0	0	0	8
2	2	2	0	0	0	0	0	0	2	0	0	0
5	5	0	5	0	5	0	0	0	0	5	0	0
6	6	0	6	0	0	6	0	0	0	0	6	0
8	8	0	0	8	0	0	8	8	0	0	0	8

one was, why couldn't we eliminate the overall force equation completely? The second one was due to the fact that the motion of each body should definitely influence the torque equations of the other bodies throughout the topological tree system, "Why don't these interrelations appear in Eqs. (37)?"

Eliminating the overall force equation does have the advantage because we then have three less force component equations to solve. The disadvantage is that the translational motion of the system is lost in this shifting process. It is really a very important part of the dynamic behavior of the multibody system. Unless the system is completely symmetric with respect to the main body, it would be better to maintain the force equation, (37a).

As far as the torque equations are concerned, it depends entirely on where the reference points are from which we take the moment. In the direct path approach, each torque equation represents physically the summation of moments about each hinge point. There are actually two ways of obtaining the torque equation about the same hinge point. The first one is to sum up the contributions from the motion of all branch bodies of this point. The other one is to sum up the contributions from the motion of all the rest of the bodies in the system. Naturally, we would like to choose the first one, which is simplier. They physical meaning of the torque equations in Eq. (47) is that we are taking moment about the instantaneous mass centers O_u^* of the subsystems of bodies, including body u. The motion of all the bodies in the topological tree system are completely "coupled." This complicates the computer calculations. Furthermore, this shifting process also complicates the coefficient matrices in the vibration equations, where they are completely uncoupled in Eq. (37c).

To illustrate the contribution from the motion of the bodies in the topological tree system to the coefficient matrices of the overall dynamic equations, we choose the 8-body system given in Table 2.

From Table 3 we find that only a few slots of the coefficient matrices require summation of the contribution from the motion of several bodies. Most of the slots require only single-body entry or zero entry. It also shows the uncoupling characteristics of the equations. When Eq. (47) is used, all the zero slots will be filled with nonzero numbers. We conclude that this shifting process certainly adds an unnecessary complication to the problem.

Comparing the three approaches presented in this analysis, the complete Newtonian approach is the most straightforward. It provides better physical understanding of the interaction among motions of the individual bodies. After obtaining the dynamic equations of a single body, there are two major steps of derivation we must go through. The first one is to transform the variables. The second one is to transform the effect of the single-body motion to the torque equations about other hinge points. If we reverse the order of steps or anxiously go into the final dynamic equations, the systematic definitions of the time-independent and time-dependent mass properties, [Eqs. (6) and (7)] and the transformed mass properties [Eqs. (32)] will not be shown clearly. From the computer programing point of view, calculating all properties of each body separately before calculating the overall properties of the system [Eqs. (38-40)] is very efficient.

In the complete Lagrangian approach, transforming the velocity is certainly easier than transforming the acceleration. We do not have to worry about the elimination of the interacting forces and torques. Carrying out the partial differentiations in Lagrange's equations is very straightforward from the mathematical point of view. However, knowing the definitions of the transformed mass properties [Eqs. (32)] beforehand makes the derivation easier.

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